Causal Interaction in High Dimension

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Interaction Effects and Causal Heterogeneity

1. Moderation
   - How do treatment effects vary across individuals?
   - Who benefits from (or is harmed by) the treatment?
   - Interaction between treatment and pre-treatment covariates

2. Causal interaction
   - What aspects of a treatment are responsible for causal effects?
   - What combination of treatments is efficacious?
   - Interaction between treatment variables

3. Individualized treatment regimes
   - What combination of treatments is optimal for a given individual?

Causal Interaction in High Dimension

- High dimension = many treatments, each having multiple levels
- A motivating application: Conjoint analysis (Hainmueller *et al.* 2014)
  - survey experiments to measure immigration preferences
  - a representative sample of 1,396 American adults
  - each respondent evaluates 5 pairs of immigirant profiles
  - `gender`^{2}, `education`^{7}, `origin`^{10}, `experience`^{4}, `plan`^{4}, `language`^{4}, `profession`^{11}, `application reason`^{3}, `prior trips`^{5}
  - Over 1 **million** treatment combinations!
  - What combinations of immigrant characteristics make them preferred?
- Too many treatment combinations $\rightsquigarrow$ Need for an effective summary
- Interaction effects play an essential role

Two Interpretations of Causal Interaction

1. Conditional effect interpretation:
    - Does the effect of one treatment change as we vary the value of another treatment?
    - Does the effect of being black change depending on whether an applicant is male or female?
    - Useful for testing moderation among treatments

2. Interactive effect interpretation:
    - Does a combination of treatments induce an *additional effect* beyond the sum of separate effects attributable to each treatment?
    - Does being a black female induce an additional effect beyond the effect of being black and that of being female?
    - Useful for finding efficacious treatment combinations in high dimension

An Illustration in the 2×2 Case

- Two binary treatments: A and B
- Potential outcomes: $Y(a, b)$ where $a, b \in \{0, 1\}$
- Conditional effect interpretation:

$$\underbrace{[Y(1,1) - Y(0,1)]}_{\text{effect of } A \text{ when } B=1} - \underbrace{[Y(1,0) - Y(0,0)]}_{\text{effect of } A \text{ when } B=0}$$

$\rightsquigarrow$ requires the specification of moderator

- Interactive effect interpretation:

$$\underbrace{[Y(1,1) - Y(0,0)]}_{\text{effect of } A \text{ and } B} - \underbrace{[Y(1,0) - Y(0,0)]}_{\text{effect of } A \text{ when } B=0} - \underbrace{[Y(0,1) - Y(0,0)]}_{\text{effect of } B \text{ when } A=0}$$

$\rightsquigarrow$ requires the specification of baseline condition

- The same quantity but two different interpretations

Difficulty of the Conventional Approach

- Lack of invariance to the baseline condition
  $\rightsquigarrow$ Inference depends on the choice of baseline condition
- 3×3 example:
  - Treatment $A \in \{a_0, a_1, a_2\}$ and Treatment $B \in \{b_0, b_1, b_2\}$
  - Regression model with the baseline condition (a_0, b_0):

$$\mathbb{E}(Y \mid A, B) \quad = \quad 1 + a_1^* + a_2^* + b_2^* + a_1^* b_2^* + 2a_2^* b_2^* + 3a_2^* b_1^*$$

  - Interaction effect for (a_2, b_2) $>$ Interaction effect for (a_1, b_2)
  - Another equivalent model with the baseline condition (a_0, b_1):

$$\mathbb{E}(Y \mid A, B) \quad = \quad 1 + a_1^* + 4a_2^* + b_2^* + a_1^* b_2^* - a_2^* b_2^* - 3a_2^* b_0^*$$

  - Interaction effect for (a_2, b_2) $<$ Interaction effect for (a_1, b_2)
  - Interaction effect for (a_2, b_1) is zero under the second model
  - All interaction effects with at least one baseline value are zero

Empirical Illustration: Lack of Invariance

- Linear regression with main effects and two-way interactions
- Baseline: *lowest* levels of job experiences and education

	Education						
Job experience	None	4th grade	8th grade	High school	Two-year college	College	Graduate
None	0 (baseline)	0	0	0	0	0	0
1–2 years	0	0.009 (0.063)	−0.019 (0.063)	−0.032 (0.063)	0.100 (0.064)	−0.044 (0.064)	−0.064 (0.063)
3–5 years	0	0.016 (0.063)	0.056 (0.064)	0.165 (0.064)	0.107 (0.064)	0.010 (0.065)	0.117 (0.063)
>5 years	0	−0.050 (0.064)	0.126 (0.064)	0.042 (0.063)	0.058 (0.064)	−0.094 (0.064)	0.015 (0.064)

The Effects of Changing the Baseline Condition

- Same linear regression but different baseline
- Baseline: *highest* levels of job experiences and education

Job experience	Education: None	4th grade	8th grade	High school	Two-year college	College	Graduate
None	0.015 (0.064)	0.065 (0.062)	−0.111 (0.064)	−0.027 (0.061)	−0.043 (0.063)	0.109 (0.063)	0
1–2 years	0.078 (0.064)	0.138 (0.062)	−0.066 (0.062)	0.006 (0.061)	0.120 (0.062)	0.129 (0.062)	0
3–5 years	−0.102 (0.062)	−0.036 (0.062)	−0.172 (0.063)	0.021 (0.062)	−0.054 (0.061)	0.002 (0.062)	0
$>$ 5 years	0	0	0	0	0	0	0 (baseline)

The Contributions of the Paper

1. Problems of the conventional approach:
   - Lack of invariance to the choice of baseline condition
   - Difficulty of interpretation for higher-order interaction
2. Solution: Average Marginal Treatment Interaction Effect
   - invariant to baseline condition
   - same, intuitive interpretation even for high dimension
   - simple estimation procedure
3. Reanalysis of the immigration survey experiment

Two-way Causal Interaction

- Two factorial treatments:

$$\begin{aligned} A &\in \mathcal{A} = \{a_0, a_1, \ldots, a_{D_A-1}\} \\ B &\in \mathcal{B} = \{b_0, b_1, \ldots, b_{D_B-1}\} \end{aligned}$$

- Assumption: Full factorial design
  1. Randomization of treatment assignment

$$\{Y(a_\ell, b_m)\}_{a_\ell \in \mathcal{A}, b_m \in \mathcal{B}} \perp\!\!\!\perp \{A, B\}$$

  2. Non-zero probability for all treatment combination

$$\Pr(A = a_\ell, B = b_m) > 0 \quad \text{for all } a_\ell \in \mathcal{A} \quad \text{and} \quad b_m \in \mathcal{B}$$

- Fractional factorial design not allowed
  1. Use a small non-zero assignment probability
  2. Focus on a subsample
  3. Combine treatments

Non-Interaction Effects of Interest

1. Average Treatment Combination Effect (ATCE):
    - Average effect of treatment combination $(A, B) = (a_\ell, b_m)$ relative to the baseline condition $(A, B) = (a_0, b_0)$

$$\tau(a_\ell, b_m; a_0, b_0) \equiv \mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_0)\}$$

    - Which treatment combination is most efficacious?

2. Average Marginal Treatment Effect (AMTE; Hainmueller et al. 2014):
    - Average effect of treatment $A = a_\ell$ relative to the baseline condition $A = a_0$ averaging over the other treatment B

$$\psi(a_\ell, a_0) \equiv \int_{\mathcal{B}} \mathbb{E}\{Y(a_\ell, B) - Y(a_0, B)\} dF(B)$$

    - Which treatment is effective on average?

The Conventional Approach to Causal Interaction

- Average Treatment Interaction Effect (ATIE):

$$\xi(a_\ell, b_m; a_0, b_0) \equiv \mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_m) - Y(a_\ell, b_0) + Y(a_0, b_0)\}$$

- Conditional effect interpretation:

$$\underbrace{\mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_m)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_m} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_0}$$

- Interactive effect interpretation:

$$\underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE}} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_0} - \underbrace{\mathbb{E}\{Y(a_0, b_m) - Y(a_0, b_0)\}}_{\text{Effect of } B = b_m \text{ when } A = a_0}$$

- Estimation: Linear regression with interaction terms

Lack of Invariance to the Baseline Condition

- Comparison between two ATIEs should not be affected by the choice of baseline conditions
- We prove that the ATIEs are *neither interval or order invariant*

- Interval invariance:

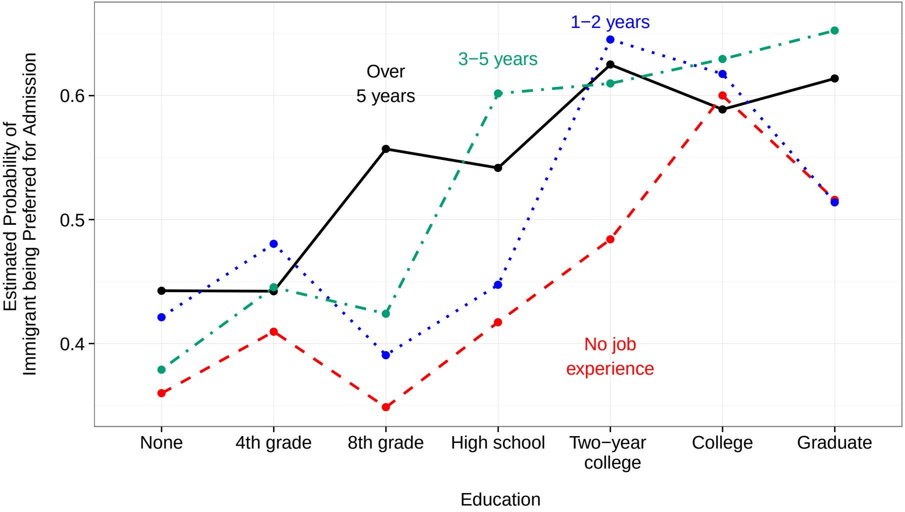
$$\begin{aligned} & \xi(a_\ell, b_m; a_0, b_0) \; - \; \xi(a_{\ell'}, b_{m'}; a_0, b_0) \\ = \; & \xi(a_\ell, b_m; a_{\tilde{\ell}}, b_{\tilde{m}}) \; - \; \xi(a_{\ell'}, b_{m'}; a_{\tilde{\ell}}, b_{\tilde{m}}), \end{aligned}$$

- Order invariance:

$$\begin{aligned} & \xi(a_\ell, b_m; a_0, b_0) \; \geq \; \xi(a_{\ell'}, b_{m'}; a_0, b_0) \\ \Longleftrightarrow \; & \xi(a_\ell, b_m; a_{\tilde{\ell}}, b_{\tilde{m}}) \; \geq \; \xi(a_{\ell'}, b_{m'}; a_{\tilde{\ell}}, b_{\tilde{m}}). \end{aligned}$$

Ineffectiveness of Interaction Plot in High Dimension

Problem: it does not plot interaction effects themselves



ATIE is Sensitive to the Choice of Baseline Condition

Job experience	Education						
	None	4th grade	8th grade	High school	Two-year college	College	Graduate
None	0	0	0	0	0	0	0
1–2 years	0	0.009	-0.019	-0.032	0.100	-0.044	-0.064
3–5 years	0	0.016	0.056	0.165	0.107	0.010	0.117
$>$ 5 years	0	-0.050	0.126	0.042	0.058	-0.094	0.015

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1–2 years	0.078	0.138	-0.066	0.006	0.120	0.129	0
3–5 years	-0.102	-0.036	-0.172	0.021	-0.054	0.002	0
>5 years	0	0	0	0	0	0	0

The New Causal Interaction Effect

- Average Marginal Treatment Interaction Effect (**AMTIE**):

$$\pi(a_\ell, b_m; a_0, b_0) \equiv \underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE of } (A,B)=(a_\ell, b_m)} - \underbrace{\psi(a_\ell, a_0)}_{\text{AMTE of } a_\ell} - \underbrace{\psi(b_m, b_0)}_{\text{AMTE of } b_m}$$

- Interactive effect interpretation: additional effect induced by $A = a_\ell$ and $B = b_m$ together beyond the separate effect of $A = a_\ell$ and that of $B = b_m$
- Compare this with ATIE:

$$\underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE}} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A=a_\ell \text{ when } B=b_0} - \underbrace{\mathbb{E}\{Y(a_0, b_m) - Y(a_0, b_0)\}}_{\text{Effect of } B=b_m \text{ when } A=a_0}$$

- We prove that the **AMTIE**s are both *interval and order invariant*
- The **AMTIE**s do depend on the distribution of treatment assignment
  1. specified by one's experimental design
  2. motivated by the target population

AMTIE is Invariant to the Choice of Baseline Condition

Job experience	None	4th grade	8th grade	High school	Two-year college	College	Graduate
				Education			
None	0	−0.004	−0.028	−0.035	−0.031	0.012	−0.010
1–2 years	−0.00	−0.001	−0.025	−0.040	0.024	−0.009	−0.044
3–5 years	−0.040	−0.019	−0.042	0.031	−0.026	−0.022	0.024
$>$ 5 years	−0.014	−0.031	0.041	−0.011	−0.021	−0.036	−0.024

AMTIE is Invariant to the Choice of Baseline Condition

Job experience	None	4th grade	8th grade	High school	Two-year college	College	Graduate
	Education						
None	0.024	0.020	−0.004	−0.011	−0.007	0.036	0.014
1–2 years	0.023	0.023	−0.001	−0.016	0.048	0.015	−0.020
3–5 years	−0.016	0.005	−0.018	0.055	−0.002	0.002	0.048
$>$ 5 years	0.010	−0.007	0.065	0.013	0.003	−0.012	0

The Relationships between the ATIE and the AMTIE

1. The **AMTIE** is a linear function of the ATIEs:

$$\begin{aligned}\pi(a_\ell, b_m; a_0, b_0) &= \xi(a_\ell, b_m; a_0, b_0) - \sum_{a \in \mathcal{A}} \Pr(A_i = a)\, \xi(a, b_m; a_0, b_0) \\ &\quad - \sum_{b \in \mathcal{B}} \Pr(B_i = b)\, \xi(a_\ell, b; a_0, b_0)\end{aligned}$$

2. The ATIE is also a linear function of the **AMTIE**s:

$$\xi(a_\ell, b_m; a_0, b_0) = \pi(a_\ell, b_m; a_0, b_0) - \pi(a_\ell, b_0; a_0, b_0) - \pi(a_0, b_m; a_0, b_0)$$

- Absence of causal interaction:
  All of the **AMTIE**s are zero if and only if all of the ATIEs are zero
- The **AMTIE**s can be estimated by first estimating the ATIEs

Higher-order Causal Interaction

- J factorial treatments: $\mathbf{T} = (T_1, \ldots, T_J)$
- Assumptions:
  1. Full factorial design
  $$Y(\mathbf{t}) \quad \perp\!\!\!\perp \quad \mathbf{T} \quad \text{and} \quad \Pr(\mathbf{T} = \mathbf{t}) \ > \ 0 \quad \text{for all } \mathbf{t}$$
  2. Independent treatment assignment
  $$T_j \quad \perp\!\!\!\perp \quad \mathbf{T}_{-j} \quad \text{for all } j$$
- Assumption 2 is not necessary for identification but considerably simplifies estimation
- We are interested in the K-way interaction where $K \leq J$
- We extend all the results for the 2-way interaction to this general case

Difficulty of Interpreting the Higher-order ATIE

- Generalize the 2-way ATIE by marginalizing the other treatments $\underline{\mathbf{T}}^{1:2}$

$$\xi_{1:2}(t_1, t_2; t_{01}, t_{02}) \equiv \int \mathbb{E}\left\{ Y(t_1, t_2, \underline{\mathbf{T}}^{1:2}) - Y(t_{01}, t_2, \underline{\mathbf{T}}^{1:2}) - Y(t_1, t_{02}, \underline{\mathbf{T}}^{1:2}) + Y(t_{01}, t_{02}, \underline{\mathbf{T}}^{1:2}) \right\} dF(\underline{\mathbf{T}}^{1:2})$$

- In the literature, the 3-way ATIE is defined as

$$\begin{aligned} & \xi_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03}) \\ \equiv \ & \underbrace{\xi_{1:2}(t_1, t_2; t_{01}, t_{02} \mid T_3 = t_3)}_{\text{2-way ATIE when } T_3 = t_3} - \underbrace{\xi_{1:2}(t_1, t_2; t_{01}, t_{02} \mid T_3 = t_{03})}_{\text{2-way ATIE when } T_3 = t_{03}} \end{aligned}$$

- Higher-order ATIEs are similarly defined sequentially
- This representation is based on the conditional effect interpretation
- Problem: the conditional effect of conditional effects!

Interactive Interpretation of the Higher-order ATIE

- We show that the higher-order ATIE also has an interactive effect interpretation
- Example: 3-way ATIE, $\xi_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})$, equals

$$
\begin{aligned}
&\underbrace{\tau_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})}_{\text{ATCE}} \\
&- \{\xi_{1:2}(t_1, t_2; t_{01}, t_{02} \mid T_3 = t_{03}) + \xi_{2:3}(t_2, t_3; t_{02}, t_{03} \mid T_1 = t_{01}) \\
&\quad + \xi_{1:3}(t_1, t_3; t_{01}, t_{03} \mid T_2 = t_{02})\} \quad \text{sum of 2-way conditional ATIEs} \\
&- \{\tau_1(t_1, t_{02}, t_{03}; t_{01}, t_{02}, t_{03}) + \tau_2(t_{01}, t_2, t_{03}; t_{01}, t_{02}, t_{03}) \\
&\quad + \tau_3(t_{01}, t_{02}, t_3; t_{01}, t_{02}, t_{03})\} \quad \text{sum of (1-way) ATCEs}
\end{aligned}
$$

- Problems:
  1. Lower-order *conditional* ATIEs rather than lower-order ATIEs are used
  2. K-way ATCE $\neq$ sum of all K-way and lower-order ATIEs
  3. (We prove) Lack of invariance to the baseline conditions

The K-way Average Marginal Treatment Interaction Effect

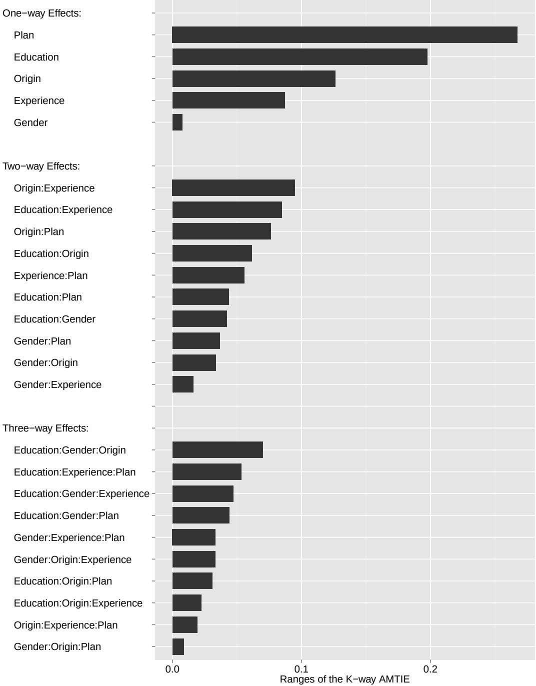
- Definition: the difference between the ATCE and the sum of lower-order **AMTIE**s
- Interactive effect interpretation
- Example: 3-way **AMTIE**, $\pi_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})$, equals

$$\underbrace{\tau_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})}_{\text{ATCE}}$$
$$- \underbrace{\left\{\pi_{1:2}(t_1, t_2; t_{01}, t_{02}) + \pi_{2:3}(t_2, t_3; t_{02}, t_{03}) + \pi_{1:3}(t_1, t_3; t_{01}, t_{03})\right\}}_{\text{sum of 2-way AMTIEs}}$$
$$- \underbrace{\left\{\psi(t_1; t_{01}) + \psi(t_2; t_{02}) + \psi(t_3; t_{03})\right\}}_{\text{sum of (1-way) AMTEs}}$$

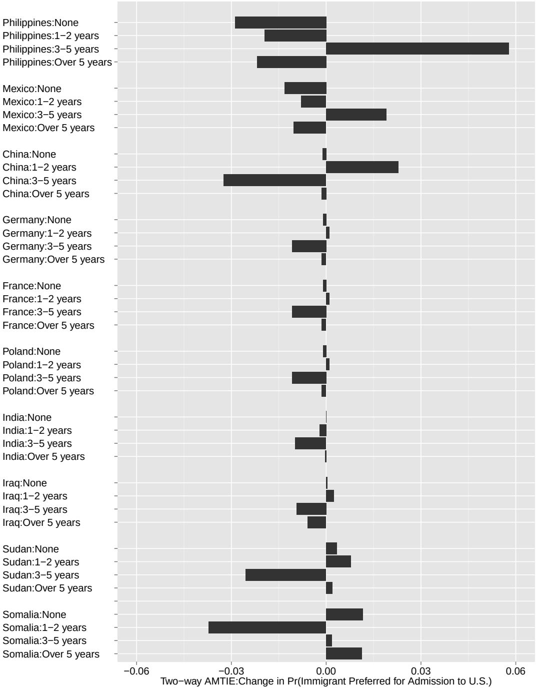
- Properties:
  1. K-way ATCE $=$ the sum of all K-way and lower-order **AMTIE**s
  2. Interval and order invariance to the baseline condition
  3. Derive the relationships between the **AMTIE**s and ATIEs for any order

Empirical Analysis of the Immigration Survey Experiment

- 5 factors (gender^2, education^7, origin^{10}, experience^4, plan^4)
  1. full factorial design assumption
  2. computational tractability
- Matched-pair conjoint analysis: randomly choose one profile
- Binary outcome: whether a profile is selected
- Model with one-way, two-way, and three-way interaction terms
- $p = 1,575$ and $n = 6,980$
- Curse of dimensionality $\Longrightarrow$ sparcity assumption
- Support vector machine with a lasso constraint (Imai & Ratkovic, 2013)
- Under-identified model that includes baseline conditions
- 99 non-zero and $1,476$ zero coefficients
- Cross-validation for selecting a tuning parameter
- `FindIt: Finding heterogeneous treatment effects`



- Range of **AMTIE**s: importance of each factor and factor interaction
- Sparcity-of-effects principle
- `gender` appears to play a significant role in three-way interactions



- Exploration of level interactions
- `origin` × `experience` interaction
- Baseline: `India`, `None`
- Only relative magnitude matters

- Little interaction for European origin
- Similar pattern for `Mexico` and `Phillipines` as well as `Sudan` and `Somalia`

Decomposing the Average Treatment Combination Effect

- Two-way effect example (`origin` $\times$ `experience`):

$$\underbrace{\tau(\texttt{Somalia, 1-2 years; India, None})}_{-3.74} \qquad (n = 168; \quad n = 155)$$

$$= \underbrace{\psi(\texttt{Somalia}; \texttt{India})}_{-5.14} + \underbrace{\psi(\texttt{1} - \texttt{2years}; \texttt{None})}_{5.12} + \underbrace{\pi(\texttt{Somalia}, \texttt{1} - \texttt{2years}; \texttt{India}, \texttt{None})}_{-3.72}$$

- Three-way examples (`education` $\times$ `gender` $\times$ `origin`):

$$\underbrace{\tau(\texttt{Graduate}, \texttt{Male}, \texttt{India}; \texttt{Graduate}, \texttt{Female}, \texttt{India})}_{7.46} \qquad (n = 52; \quad n = 40)$$

$$= \underbrace{\psi(\texttt{Male}; \texttt{Female})}_{-0.77} + \underbrace{\pi(\texttt{Graduate}, \texttt{Male}; \texttt{Graduate}, \texttt{Female})}_{-0.34}$$

$$+ \underbrace{\pi(\texttt{Male}, \texttt{India}; \texttt{Female}, \texttt{India})}_{1.56} + \underbrace{\pi(\texttt{Graduate}, \texttt{Male}, \texttt{India}; \texttt{Graduate}, \texttt{Female}, \texttt{India})}_{7.01}$$

$$
\underbrace{\tau(\texttt{High school, Male, Germany; High school, Female, Germany})}_{-11.52}
$$

$$
(n = 41; \quad n = 56)
$$

$$
= \underbrace{\psi(\texttt{Male}; \texttt{Female})}_{-0.77} + \underbrace{\pi(\texttt{High school, Male; High school, Female})}_{-0.67}
$$

$$
+ \underbrace{\pi(\texttt{Male}, \texttt{Germany}; \texttt{Female}, \texttt{Germany})}_{-3.34}
$$

$$
+ \underbrace{\pi(\texttt{High school, Male,Germany; High school, Female,Germany})}_{-6.74}.
$$

Concluding Remarks

- Interaction effects play an essential role in causal heterogeneity
  1. moderation
  2. causal interaction
- Two interpretations of causal interaction
  1. conditional effect interpretation (problematic in high dimension)
  2. interactive effect interpretation
- Average Marginal Treatment Interaction Effect
  1. interactive effect in high-dimension
  2. invariant to baseline condition
  3. enables effect decomposition
  4. $\rightsquigarrow$ effective analysis of interactions in high-dimension
- Estimation challenges in high dimension
  1. group lasso, hierarchical interaction
  2. post-selection inference

References

1. Imai, Kosuke and Marc Ratkovic. (2013). "Estimating Treatment Effect Heterogeneity in Randomized Program Evaluation." *Annals of Applied Statistics*, Vol. 7, No. 1 (March), pp. 443–470.

2. Egami, Naoki and Kosuke Imai. (2015). "Causal Interaction in High Dimension." Working Paper available at `http://imai.princeton.edu/research/int.html`

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